

# INSIDE-OUTSIDE ESTIMATION MEETS DYNAMIC EM

Detlef Prescher

DFKI Language Technology Lab

Stuhlsatzenhausweg 3, 66123 Saarbrücken, Germany

prescher@dfki.de

## Abstract

We briefly review the inside-outside and EM algorithm for probabilistic context-free grammars. As a result, we formally prove that inside-outside estimation is a dynamic-programming variant of EM. This is interesting in its own right, but even more when considered in a theoretical context since the well-known convergence behavior of inside-outside estimation has been confirmed by many experiments but apparently has never been formally proved. However, being a version of EM, inside-outside estimation also inherits the good convergence behavior of EM. Therefore, the as yet imperfect line of argumentation can be transformed into a coherent proof.

## 1 Inside-Outside Estimation

The modern **inside-outside algorithm** was introduced by [4] who reviewed an algorithm proposed by [1] and extended it to an iterative training method for probabilistic context-free grammars enabling the use of unrestricted free text. In the following,  $y_1 \dots y_N$  are numbered (but unannotated) sentences.

**Definition:** Inside-outside re-estimation formulas for probabilistic context-free grammars in Chomsky normal form are given by (see [4], but see also [1] for the special case  $N = 1$ ):

$$\hat{p}(A \rightarrow a) := \frac{\sum_{w=y_1}^{y_N} C_w(A \rightarrow a)}{\sum_{w=y_1}^{y_N} C_w(A)}, \text{ and } \hat{p}(A \rightarrow BC) := \frac{\sum_{w=y_1}^{y_N} C_w(A \rightarrow BC)}{\sum_{w=y_1}^{y_N} C_w(A)}.$$

The key variables of this definition are so-called **category** and **rule counts**:  $C_w(A) := \frac{1}{P} \sum_{s=1}^n \sum_{t=s}^n e(s, t, A) \cdot f(s, t, A)$ ,  $C_w(A \rightarrow a) := \frac{1}{P} \sum_{1 \leq t \leq n, w_t=a} e(t, t, A) \cdot f(t, t, A)$ , and  $C_w(A \rightarrow BC) := \frac{1}{P} \sum_{s=1}^{n-1} \sum_{t=s+1}^n \sum_{r=s}^{t-1} p(A \rightarrow BC) e(s, r, B) e(r+1, t, C) f(s, t, A)$  which are computed for each sentence  $w := w_1 \dots w_n$  with so-called inside and outside probabilities: An **inside probability** is defined as the probability of category  $A$  generating observations  $w_s \dots w_t$ , i.e.  $e(s, t, A) := p(A \Rightarrow^* w_s \dots w_t)$ . In determining a recursive procedure for calculating  $e$ , two cases must be considered:

- ( $s = t$ ): Only one observation is emitted and therefore a rule of the form  $A \rightarrow w_s$  applies:  $e(s, s, A) = p(A \rightarrow w_s)$ , if  $(A \rightarrow w_s) \in G$  (and 0, otherwise).
- ( $s < t$ ): In this case we know that rules of the form  $A \rightarrow BC$  must apply since more than one observation is involved. Thus,  $e(s, t, A)$  can be expressed as follows:  $e(s, t, A) = \sum_{(A \rightarrow BC) \in G} \sum_{r=s}^{t-1} p(A \rightarrow BC) \cdot e(s, r, B) \cdot e(r+1, t, C)$ .

The quantity  $e$  can therefore be computed recursively by determining  $e$  for all sequences of length 1, then 2, and so on. The sentence probability  $P := p(S \Rightarrow^* w)$  is a special inside probability. The **outside probabilities** are defined as follows:  $f(s, t, A) = p(S \Rightarrow^* w_1 \dots w_{s-1} A w_{t+1} \dots w_n)$ .

The quantity  $f(s, t, A)$  may be thought of as the probability that  $A$  is generated in the re-write process and that the strings not dominated by it are  $w_1 \dots w_{s-1}$  to the left and  $w_{t+1} \dots w_n$  to the right. In this case, the non-terminal  $A$  could be one of two possible settings  $C \rightarrow B A$  or  $C \rightarrow A B$ , hence:  $f(s, t, A) = \sum_{B, C \in G} \left( \sum_{r=1}^{s-1} f(r, t, C) \cdot p(C \rightarrow BA) \cdot e(r, s-1, B) + \sum_{r=t+1}^n f(s, r, C) \cdot p(C \rightarrow AB) \cdot e(t+1, r, B) \right)$  and  $f(s, t, A) = \begin{cases} 1 & \text{if } A = S \\ 0 & \text{else} \end{cases}$ . After the inside

probabilities have been computed bottom-up, the outside probabilities can therefore be computed top-down. Unfortunately, no convergence proofs of inside-outside estimation were given by [1] and [4].

## 2 EM for Probabilistic Context-Free Grammars

The EM algorithm was introduced by [3] as iterative maximum likelihood estimation for parameterized probability models  $p(y)$  using a sample  $\tilde{p}(y)$  of **incomplete data types**  $y$  which are defined via a **symbolic analyzer**  $X(y)$  dealing with **complete data types**  $x$ . It is known, that EM generalizes ordinary maximum likelihood estimation and monotonically increases the log-likelihood  $L(p) := \sum_y \tilde{p}(y) \cdot \log \sum_{x \in X(y)} p(x)$ . Furthermore, the limit point of a convergent parameter sequence is a stationary point (i.e. local minimum, saddle point or maximum) of the log likelihood [3]. Moreover, both the parameter sequence and the associated sequence of log likelihood values converge (in some cases to local maxima), if some weak conditions are fulfilled [6].

Applying EM to probabilistic context-free grammars, the **grammatical sentences**  $y$  are viewed as incomplete and their **syntax trees**  $x$  as complete. The required symbolic analyzer is given by a **parser** computing all trees  $x \in \mathcal{T}(y)$  for a sentence  $y$ . Via these non-probabilistic EM components, the probability model for the sentences is defined as  $p(y) := \sum_{x \in \mathcal{T}(y)} p(x) := \sum_{x \in \mathcal{T}(y)} \prod_r p(r)^{f_r(x)}$ , where  $f_r(x)$  is the frequency of rule  $r$  occurring in  $x$ , and parameterization is given by **rule probabilities**  $p(r)$ . The key variables of EM re-estimation are **conditional expected frequencies** (relying on the conditional probability  $p(x|y) := \frac{p(x)}{p(y)}$ ) for rules  $r$  and categories  $A$ :  $p(\cdot|y)[f_r] := \sum_{x \in \mathcal{T}(y)} p(x|y) \cdot f_r(x)$  and  $p(\cdot|y)[f_A] := \sum_{x \in \mathcal{T}(y)} p(x|y) \cdot f_A(x)$ , where  $f_A(x) := \sum_{r \in G_A} f_r(x)$  is the frequency of category  $A$  occurring in  $x$ , and  $G_A$  is the set of grammar rules with left-hand side  $A$ . See e.g. [5]:

**Lemma:** EM re-estimation formulas for probabilistic context-free grammars are given by:

$$\hat{p}(r) = \frac{\tilde{p}[p(\cdot|y)[f_r]]}{\tilde{p}[p(\cdot|y)[f_A]]} = \frac{\sum_y \tilde{p}(y) \cdot p(\cdot|y)[f_r]}{\sum_y \tilde{p}(y) \cdot p(\cdot|y)[f_A]} \quad (r \in G, A = \text{lhs}(r)) .$$

## 3 Inside-Outside as Dynamic EM

In this section, the well-known convergence properties of the inside-outside algorithm, which have been unfortunately omitted in the original literature ([1], [4]), will be formally proven. For this purpose, we will show that the inside-outside algorithm is a dynamic-programming variant of the EM algorithm for context-free grammars. This property is also well-known in stochastic linguistics, but to the best of our knowledge all mentioned properties have not been formally proven till now.

**Theorem:** For a context-free grammar in Chomsky normal form, let  $\hat{p}(r)$  be re-estimated rule probabilities resulting from one single step of the inside-outside algorithm using the current rule probabilities  $p(r)$ . Then: (i) The log likelihood  $L(\cdot)$  of the training corpus increases monotonically, i.e.  $L(\hat{p}) \geq L(p)$ . (ii) The limit points of a sequence of re-estimated probabilities are stationary points (i.e. maxima, minima or saddle points) of the log likelihood function. (iii) The inside-outside

algorithm is a dynamic-programming variant of the EM algorithm, i.e.  $\hat{p}(r)$  corresponds to  $\hat{p}_{EM}(r)$  resulting from one single EM iteration (using also  $p(r)$  as current rule probabilities).

**Proof:** (i) and (ii) follow using both (iii) and the convergence properties of EM. (iii): The empirical distribution of the sentences is defined as  $\tilde{p}(y) = \frac{f(y)}{N}$ , where  $f(y)$  is the frequency of  $y$  occurring in the corpus  $y_1 \dots y_N$ . Thus, for each rule  $r$  with left-hand side  $A$ :  $\hat{p}_{EM}(r) = \frac{\sum_{y=y_1}^{y_N} \sum_{x \in \mathcal{T}(y)} p(x|y) \cdot f_r(x)}{\sum_{y=y_1}^{y_N} \sum_{x \in \mathcal{T}(y)} p(x|y) \cdot f_A(x)}$ . Comparing these formulas with the re-estimation formulas presented by [4], it follows  $\hat{p}_{EM}(r) = \hat{p}(r)$ , if for each sentence  $y$ , for each rule  $r$  and each category  $A$  the following propositions can be shown:

$$C_y(r) = \sum_{x \in \mathcal{T}(y)} p(x|y) \cdot f_r(x), \text{ and } C_y(A) = \sum_{x \in \mathcal{T}(y)} p(x|y) \cdot f_A(x) .$$

This is the goal of the rest of the proof, which we split in two lemmas. The first lemma is probably due to [2], where corresponding formulas are used, but not explicitly proven, to present inside-outside estimation. The lemma says that category counts can be computed by summing certain rule counts.

**Lemma:**  $C_y(A) = \sum_{r \in G_A} C_y(r)$  for each sentence  $y$  and each category  $A$ .

**Proof:** Assuming Chomsky normal form, and  $y = w_1 \dots w_n$ :

$$\begin{aligned} \sum_{r \in G_A} C_y(r) &= \sum_a C_y(A \rightarrow a) + \sum_{B, C \in G} C_y(A \rightarrow B C) \\ &= \sum_a \frac{1}{P} \sum_{1 \leq t \leq n, w_t = a} e(t, t, A) f(t, t, A) \\ &\quad + \sum_{B, C \in G} \frac{1}{P} \sum_{s=1}^{n-1} \sum_{t=s+1}^n \sum_{r=s}^{t-1} p(A \rightarrow BC) e(s, r, B) e(r+1, t, C) f(s, t, A) \\ &= \frac{1}{P} \left( \sum_{1 \leq t \leq n} e(t, t, A) f(t, t, A) \right. \\ &\quad \left. + \sum_{s=1}^{n-1} \sum_{t=s+1}^n f(s, t, A) \sum_{B, C \in G} \sum_{r=s}^{t-1} p(A \rightarrow BC) e(s, r, B) e(r+1, t, C) \right) \\ &= \frac{1}{P} \left( \sum_{1 \leq t \leq n} e(t, t, A) f(t, t, A) + \sum_{s=1}^{n-1} \sum_{t=s+1}^n f(s, t, A) e(s, t, A) \right) \\ &= \frac{1}{P} \sum_{1 \leq s \leq t \leq n} e(s, t, A) f(s, t, A) = C_y(A) . \end{aligned}$$

In the fourth equation, we used the recursion formula of the inside probabilities. **q.e.d.**

It follows that the desired identities for the category counts can be calculated (by summation over all rules with the same left-hand side) using the identities for the rule counts, since  $C_y(A) = \sum_{A \rightarrow \alpha} C_y(A \rightarrow \alpha)$ , and per definition  $f_A(x) = \sum_{A \rightarrow \alpha} f_{A \rightarrow \alpha}(x)$ . Thus, the proof of the theorem is completed, as once as the following central lemma has been proven. It states that the counts of the inside-outside algorithm can be identified with the expected rule frequencies of the EM algorithm.

**Lemma:** For each sentence  $y$  and each rule  $r$ :  $C_y(r) = \sum_{x \in \mathcal{T}(y)} p(x|y) \cdot f_r(x) = p(\cdot|y) [f_r]$ .

**Proof:** The second equation is simply the definition of the expectation. Assuming Chomsky normal form, two cases must be considered. First, the rule has the form  $A \rightarrow B C$ :

For a given sentence  $y = w_1 \dots w_n$  and given three **spans**  $(s, r, B)$ ,  $(r+1, t, C)$ ,  $(s, t, A)$  with  $1 \leq s \leq r < t \leq n$ , let  $X_{(s,t,A)(s,r,B)(r+1,t,C)}$  be the **parse forest** corresponding to the following **derivation**:  $S \Rightarrow^* w_1 \dots w_{s-1} A w_{t+1} \dots w_n \Rightarrow w_1 \dots w_{s-1} B C w_{t+1} \dots w_n \Rightarrow^* w_1 \dots w_r C w_{t+1} \dots w_n \Rightarrow^*$

$w_1 \dots w_n$ . Let  $f_{(s,t,A)(s,r,B)(r+1,t,C)}(x) := \begin{cases} 1 & \text{if } x \in X_{(s,t,A)(s,r,B)(r+1,t,C)} \\ 0 & \text{else} \end{cases}$  be the **characteristic function** interpreting  $X_{(s,t,A)(s,r,B)(r+1,t,C)}$  as a simple subset of the set of all possible syntax trees  $\mathcal{T}(y)$  of the sentence  $y$ . Thus, the frequency  $f_{A \rightarrow BC}(x)$  of the rule  $A \rightarrow BC$  occurring in the syntax tree  $x \in \mathcal{T}(y)$  can be computed as follows:

$$f_{A \rightarrow BC}(x) = \sum_{1 \leq s \leq r < t \leq n} f_{(s,t,A)(s,r,B)(r+1,t,C)}(x) .$$

Using the **linear properties of the expected frequencies**  $p(\cdot|y) [ \cdot ]$ , it follows:

$$\begin{aligned} p(\cdot|y) [ f_{A \rightarrow BC} ] &= p(\cdot|y) \left[ \sum_{1 \leq s \leq r < t \leq n} f_{(s,t,A)(s,r,B)(r+1,t,C)} \right] \\ &= \sum_{1 \leq s \leq r < t \leq n} p(\cdot|y) [ f_{(s,t,A)(s,r,B)(r+1,t,C)} ] \\ &= \sum_{1 \leq s \leq r < t \leq n} \sum_{x \in \mathcal{T}(y)} p(x|y) \cdot f_{(s,t,A)(s,r,B)(r+1,t,C)}(x) \\ &= \frac{1}{p(y)} \sum_{1 \leq s \leq r < t \leq n} \sum_{x \in \mathcal{T}(y)} p(x) \cdot f_{(s,t,A)(s,r,B)(r+1,t,C)}(x) \\ &= \frac{1}{p(y)} \sum_{1 \leq s \leq r < t \leq n} \sum_{x \in X_{(s,t,A)(s,r,B)(r+1,t,C)}} p(x) \\ &= \frac{1}{p(y)} \sum_{1 \leq s \leq r < t \leq n} p(X_{(s,t,A)(s,r,B)(r+1,t,C)}) \\ &= \frac{1}{P} \sum_{1 \leq s \leq r < t \leq n} f(s,t,A) \cdot p(A \rightarrow BC) \cdot e(s,r,B) \cdot e(r+1,t,C) \\ &= C_y(A \rightarrow BC) . \end{aligned}$$

The second case, for rules of the form  $A \rightarrow a$ , follows analogously with spans  $(s, s, A)$  and  $(s, s, a)$ . Here, the details are omitted, but see [5] **q.e.d.**

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